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ANALYSIS OF TRANSIENT STRUCTURAL RESPONSE BY THE METHOD OF CHARACTERISTICS

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SUMMARY

A brief summary of a unified approach to the transient response of structures obtained by the method of characteristics is presented. Comments are made on the types of governing partial differential equations, the stability in numerical calculation, the extrapolation technique, and the relation between continuous and discrete systems. The merits and limitations of the method of characteristics, as compared to other methods, are also discussed.

I. INTRODUCTION

With the availability of high speed digital computers, several numerical methods are being developed for the study of the transient response of structures. Among these are the finite-difference method, the finite-element method and the method of characteristics. In this paper, recent developments of the method of characteristics are summarized. Comparisons between the method of characteristics and the other methods are presented. A special emphasis is placed on the importance of totally hyperbolic differential equations for solving transient structural problems. It is shown that even though non-hyperbolic equations are sometimes used by structural engineers for vibration and steady state dynamic problems, they are not applicable for transient problems. The difference between a continuous structure and its equivalent discrete system is demonstrated. The numerical stability and the technique of extrapolation are also discussed.

II. METHOD OF CHARACTERISTICS

Motions of many types of structures that are governed by equations involving only one space variable can be analyzed by the unified approach of the method of characteristics introduced in Ref. 1. A Timoshenko beam is one typical structure amenable to this method of analysis. According to the theory of elasticity, a beam is a three-dimensional structure; its exact stress and displacement distributions are very difficult to obtain. However, if the approximate assumption that a plane cross-section remains plane is made, the governing equations simplify into two second order partial differential equations with the deflection and rotation as the dependent variables, and the axial coordinate and time as the independent variables. For a constant cross-section beam, these equations are as follows (Ref. 2).

$$\begin{aligned} \psi'' - \frac{1}{c_b^2}\ddot{\psi} &= a\,y' + b\,\psi \\ y'' - \frac{1}{c_s^2}\ddot{y} &= \psi' \end{aligned} \qquad (1)$$

where prime and dots represent spacial and time derivatives, respectively; $c_b = (E/\rho)^{1/2}$ = bar velocity; $c_s = k(G/\rho)^{1/2}$ = shear velocity; a and b are constants depending on the beam properties. Equations (1) are totally hyperbolic. Therefore, the first of (1) may be transformed into a new coordinate system α_1, β_1, with the co-tangent of the angle between the α_1-axis and x-axis equal to c_b, (Fig. 1). In the α_1, β_1 coordinates, the first of (1) becomes

$$\begin{aligned} \frac{\partial\dot{\psi}}{\partial\alpha_1} - c_b\frac{\partial\psi'}{\partial\alpha_1} + c_b\,(a\,y' + b\,\psi)\frac{\partial x}{\partial\alpha_1} &= 0 \\ \frac{\partial\dot{\psi}}{\partial\beta_1} + c_b\frac{\partial\psi'}{\partial\beta_1} - c_b\,(a\,y' + b\,\psi)\frac{\partial x}{\partial\beta_1} &= 0 \end{aligned} \qquad (2)$$

If $\dot{\psi}$, ψ', $\dot{y}$ and y' are considered as the dependent variables, then the first of (2) contains derivatives with respect to α_1 only, while the second of (2) contains derivatives with respect to β_1 only. Thus, along α_1 and β_1, the original second order partial differential equations become first order "ordinary" differential equations, a simplification which is very convenient for numerical integration.

Similarly, the second of (1) may be transformed into

$$\begin{aligned} \frac{\partial \dot{y}}{\partial \alpha_2} - c_s \frac{\partial y'}{\partial \alpha_2} + c_s \frac{\partial x}{\partial \alpha_2} &= 0 \\ \frac{\partial \dot{y}}{\partial \beta_2} + c_s \frac{\partial y'}{\partial \beta_2} - c_s \frac{\partial x}{\partial \beta_2} &= 0 \end{aligned} \tag{3}$$

where α_2, β_2 are coordinates shown in Figure 1. Equations (2) and (3) can be changed readily into finite-difference form. These finite difference equations, together with the equations,

$$\begin{aligned} d\psi &= \psi' \ dx + \dot{\psi} \ dt \\ dy &= y' \ dx + \dot{y} \ dt \end{aligned} \tag{4}$$

constitute a system of six equations for the six variables $\dot{\psi}$, ψ', ψ, $\dot{y}$, y', and y. If the values of these six variables at points 2, 3 and 4 of Figure 1 are known, their values at point 1 may be obtained from these six equations. (The values at points 5 and 6 may be calculated by interpolation). It can also be shown that discontinuities in the derivatives of ψ and y can only exist across lines parallel to the characteristic coordinates α_1, α_2, β_1 and β_2.

For the analysis of any other types of structures whose governing equations involve only one space variable the unified method of characteristics of Ref. 1 is applicable. These include different types of bars, sheets, plates, shells and springs. Two recent articles, one concerning sandwich spherical caps (Ref. 3), and the other concerning coupled bending-torsion of beams (Ref. 4), contain equations of the type treated in Ref. 1. In cases where two space variables are involved, modal analysis may be utilized to remove one of the space coordinates, while the remaining portion of the equations in terms of the other space variable can still be treated by the method of characteristics. An example of this case is a cylindrical shell subjected to a non-symmetrical axial impact. The displacement variables may be expanded into Fourier components in the circumferential coordinate; the resulting governing equations contain only one spatial independent variable (the axial coordinate) and can be treated by the method of Ref. 1.

The Timoshenko beam involves two governing equations of the second order. In other problems, the number of governing second order equations can be four, five, and even six. While the characteristic directions, the governing equations in the characteristic coordinates, as well as the equations governing the propagation of discontinuities are derived in Ref. 1 for any number of equations, the scheme for numerical integration has been developed for those cases containing only two distinct wave velocities. The symmetrical response of shells involves three equations, but only two distinct wave velocities, and thus may be calculated by the present scheme. Numerical procedures for problems involving three distinct wave velocities are currently being developed.

Since the method of characteristics is essentially numerical, it can treat equations of variable coefficients, or structures of variable mass and stiffness distributions. The wave propagation in a nonhomogeneous structure is solved by this method in Ref. 5.

Another advantage of using the unified equations of Ref. 1 is that very often two different structural problems have the same governing equations. For instance, the equations governing a correctly formulated membrane theory for cylindrical shells, Ref. 6, have the same form as the Timoshenko beam equations, (1). Therefore, all solutions for the Timoshenko beam are also solutions to the membrane shell.

By combining the two second order equations of (1), a fourth order equation may be obtained as follows, (Ref. 2),

$$y'''' - \left(\frac{1}{c_s^2} + \frac{1}{c_b^2}\right) \ddot{y}'' + \frac{1}{c_b^2 c_s^2} \ddddot{y} = - \frac{b}{c_s^2} \ddot{y} \tag{5}$$

where the fact that a = -b has been utilized. Alternatively, (1) may be decomposed into four first order equations (Ref. 7). Comparison of these three systems (one fourth order equation, two second order equations, and four first order equations) shows that the representation with two second order equations has some distinguished advantages. The wave velocities associated with each of the variables appear explicitly in the second order equations. The factors governing the propagation of discontinuities also appear explicitly in (1). In using either the first order or the fourth order equations, the hyperbolicity of the system is not immediately apparent. The same comments can also be applied to equations governing other types of structures. For instance, the cylindrical shell theory including rotary inertia and shear effects yields three second order equations, as follows (Ref. 6),

$$u'' - \frac{1}{c_p^2}\,\ddot{u} = \beta_{13}\,w'$$

$$\psi'' - \frac{1}{c_p^2}\,\ddot{\psi} = \alpha_{22}\,\psi + \beta_{23}\,w' \tag{6}$$

$$w'' - \frac{1}{c_s^2}\,\ddot{w} = \alpha_{33}\,w + \beta_{31}\,u' + \beta_{32}\,\psi'$$

From this set of equations, it is immediately evident that the system is totally hyperbolic, and with two distinct wave velocities c_p and c_s. Combining (6) into one sixth order equation, or decomposing it into six first order equations would conceal these features and would make the application of the method of characteristics more difficult.

III. HYPERBOLIC DIFFERENTIAL EQUATIONS

Another point of interest is the importance of the hyperbolic nature of the governing equations in transient structural problems. From a structural point of view, it takes a finite, though small, time for any disturbance to transmit through a structure. This observation is in agreement with the theory of elasticity where all disturbances, or excitations, are propagated at either one of the two wave velocities, the dilatational velocity or the equivoluminal velocity (Ref. 8). In deriving simple practical equations for structures, it is customary to make approximating assumptions, or to neglect certain effects of small magnitude. It must be kept in mind that for transient response purposes, the derived governing equations must be totally hyperbolic; if not, their transient response is either meaningless or not obtainable. Take the case of Timoshenko beam equations as an example. Equations (1) are totally hyperbolic; any suddenly applied disturbance in ψ, or in moment, propagates at the bar velocity, c_b, while a disturbance in y, or in shear, propagates at the shear velocity, c_s. The contribution of rotary inertia appears in these equations as the term $\ddot{\psi}$. If this rotary inertia term is neglected, the first of (1) is no longer hyperbolic, but parabolic. Any disturbance in ψ, or in moment, is immediately felt at infinity, as evidenced by the fact that setting c_b equal to infinity is equivalent to dropping the $\ddot{\psi}$ term. In the classical Bernoulli-Euler beam theory, where both rotary inertia and shear effects are neglected, the governing equation

$$y'''' + \frac{\rho A}{EI} \ddot{y} = 0 \tag{7}$$

is not hyperbolic, but totally parabolic; therefore, cannot be used for transient problems. (Applying the method of characteristics to (7), we obtain four degenerated characteristics, all in the direction $dt = 0$).

In vibration and harmonic wave studies, (7) has often been used. In these cases, "steady state," rather than transient, solutions are involved; therefore, (7) can still produce "meaningful," even though less accurate, results. As mentioned before, the Timoshenko equations are also approximate, and have their limitations. However, these approximations do not alter the hyperbolic nature of the exact elasticity equations. It is well known that the natural frequencies of beams according to Timoshenko theory are more accurate than those according to Bernoulli-Euler theory (Refs. 9, 10). Consequently, the Timoshenko equations for a beam are essential in transient analysis; they are also the most suitable for vibration and steady state problems for their accuracy.

The same situation exists in the membrane theory for shells. In solving transient shell problems, many investigators merely add inertia terms to a set of existing static membrane equations, without checking the nature of the resulting equations, (Ref. 11). In the pure static case, the shear effect in a membrane shell may be neglected without any harm. However, in the corresponding dynamic case, if shear is not included, the equations are again not totally hyperbolic. A set of correctly formulated membrane equations for cylindrical shells may be obtained from (6) by dropping the second equation, and the ψ term in the third, which results in

$$\begin{aligned} w'' - \frac{1}{c_s^2} \ddot{w} &= \beta_{31} u' + \alpha_{33} w \\ u'' - \frac{1}{c_p^2} \ddot{u} &= \beta_{13} w' \end{aligned} \tag{8}$$

This is a set of totally hyperbolic equations. Neglecting the shear effect, as is commonly done, is equivalent to dropping the w'' term. Regardless of how small and neglegible the shear effect is in the static case, it must be retained in the dynamic equations. Furthermore, the form of (8) is already simple; dropping of w'' term seems to have no justification from simplicity point of view.

As mentioned before, the governing equations of a Timoshenko beam, (1), are the same as the correctly formulated membrane equations, (8) except for the difference in coefficients. Typical results calculated by the method of characteristics for a cylindrical shell subjected to an end step axial velocity are shown in Figure 2. The shear stress distribution from both the general bending theory and the simplified membrane theory are given.

IV. STABILITY OF NUMERICAL METHODS

Since the method of characteristics is essentially a numerical method, the question of stability is of extreme importance. In problems involving one space variable, such as those treated in Refs. 1 and 12, the numerical scheme adopted is inherently stable. For two space variable transient problems, numerical schemes are not always stable (Ref. 13). The question of stability must be established for each problem separately, and is usually very difficult. This section is intended to give only a brief introduction to stability and a demonstration of its importance.

Let $u(x,t)$ be the exact solution of a transient problem, u_j^n be the numerical solution of the same problem at time $t = n\Delta t$ and position $x = j\Delta x$ where Δt and Δx are the mesh sizes used in the numerical calculation. Then the error, e, is given by

$$e = u(x,t) - u_j^n \tag{9}$$

Stability may be defined in two different ways, (Ref. 14). The first states that a numerical scheme is stable if e is bounded as n approaches infinity for fixed Δx and Δt. The second definition stipulates that a numerical scheme is stable if e is bounded as Δt and Δx approach zero, and n approaches infinity, for a fixed value of $t = n\Delta t$. In either case, it is necessary to observe a "stability criterion" in order to prevent errors from amplifying so much as to make the calculations meaningless. The "stability criterion" usually amounts to a restriction on the permissible size of Δt in terms of the sizes of spacial increments. Lack of adherence to the criterion produces symptoms of instability within a small number of cycles.

In one space variable problems, the method of characteristics is stable because it always adheres to the Courant-Friedrich-Lewy stability criterion

$$\frac{c\Delta t}{\Delta x} \leq 1 \qquad (10)$$

where c is the largest of the wave velocities. For two space variable problems, exact form of necessary and sufficient stability criteria is hard to establish. The following example demonstrates the occurence and symptom of instability in a specific numerical case.

The problem treated is a step stress input applied at the interior surface of a spherical cavity in an elastic medium. In terms of the scalar displacement potential, the governing equation is

$$\nabla^2\phi - \frac{1}{c^2}\ddot{\phi} = 0 \qquad (11)$$

where c is the dilatational velocity. In order to develop the two space variable method of characteristics, this spherically symmetrical problem is treated purposely by cylindrical coordinates r and z. At a certain time after the input load is applied, the exact solution by Sharpe, Ref. 15, is taken as the initial value for our two-dimensional initial value problem. Since only the dilatational wave is involved, only one characteristic cone extends from each point. Four bicharacteristics are chosen, and the corresponding characteristic equations are written. By following a numerical scheme similar to that used by Butler (Ref. 16), the displacement and stress at points on succeeding constant time planes are calculated. In all the calculations equal mesh sizes are used for Δr and Δz ($\Delta r = \Delta z$) but the ratio $\Delta t/\Delta r$ is varied. Figure 3 shows a plot of the cylindrical radial component of displacement at a point r = z - 1.31, as a function of time. For a ratio of $\Delta t/\Delta r = 0.6$ the calculation is stable, producing very smooth accurate results. For a ratio $\Delta t/\Delta r = 0.8$, the calculation is unstable; after approximately fifteen time steps, the value from each successive calculation oscillates violently, and is apparently meaningless.

V. ERROR ANALYSIS AND EXTRAPOLATION

According to the definition of stability, adherence to the stability criteria ensures the convergence of the numerical solution to the true solution, as Δt and Δx approach zero. For practical purposes, the rate of convergence is also of great interest. This rate depends primarily on the truncation error, which is the error introduced by approximating the differentials by finite differences. If the type of truncation error of a given problem is known, an extrapolation technique can be employed to achieve a high degree of accuracy with a small amount of calculation. The error of a numerical calculation, as defined in (9), is said to be of h^2-type if it can be expressed in the form

$$e = u - u_j^n = \phi_1 h^2 + \phi_2 h^4 + \phi_3 h^6 + \ldots + \phi_i h^{2i} + \ldots \quad (12)$$

where h is the mesh size (Δt, or Δx), and the ϕ_i's are quantities that are independent of the mesh size and are dependent only upon the x,t-location of the point at which the function of u_j^n is being evaluated. If two values, u_1 and u_2, at a given point are calculated (u_1 corresponding to a mesh size h_1 and u_2 corresponding to h_2) then we may write (12) twice in truncated form,

$$\begin{aligned} u &= u_1 + \phi_1 h_1^2 \\ u &= u_2 + \phi_1 h_2^2 \end{aligned} \quad (13)$$

Elimination of ϕ_1 from these equations gives the extrapolated value of u,

$$u = (h_2^2 u_1 - h_1^2 u_2)/(h_2^2 - h_1^2) \quad (14)$$

This formula is called the h^2-type two point extrapolation. Similarly, if calculations with three different mesh sizes h_1, h_2, and h_3 are performed, a three-point h^2-type extrapolation formula may be written from the three equations obtained from (12) by truncating terms containing h^6 and higher order.

The principle of extrapolation can be best demonstrated by graphical means. Let us define the percent error as the difference between the true value and the calculated value, divided by the true value. The three points on Figure 4(a) represent the percent error for three calculations with mesh sizes h = 1, 3/4, and 1/2 respectively. The intersection of the straight line joining the 1, and 3/4 error points, with the vertical axis, gives the two point extrapolated value of (14). Similarly, a quadratic curve passing through all three points gives the three-point h^2- type extrapolation.

If the error is of h-type, or

$$e = \psi_1 h + \psi_2 h^2 + \psi_3 h^3 + \ldots\ldots + \psi_i h^i \ldots\ldots \qquad (15)$$

similar h-type extrapolation formulas may be devised. Obviously, for a given problem, the error cannot be of both h- and h^2-types. The errors given in Figure 4 are close to h^2-type; therefore, all h^2-type extrapolations improve the results (Figure 4a), but h-type extrapolations do not (Figure 4b).

As an illustrative example, we shall consider again the spherical dilatational wave problem solved exactly by Sharpe (Ref. 15). The same problem is now solved by the numerical method of characteristics of reference 1, with three different mesh sizes corresponding to h = 1, 1/2, and 1/4 respectively. Fifteen decimal digits were used both for the evaluation of Sharpe's exact solution, and for the calculations by the method of characteristics. The curves in Figure 5 labeled h = 1, h = 1/2, and h = 1/4, are the percent error in radial stress, at 3.2 μsec. after loading, for the three mesh sizes. The curve $h^2(1, 1/2)$ is the extrapolated value, according to h^2-type extrapolation, from those values with h = 1 and 1/2; the curve $h^2(1, 1/2, 1/4)$ is the absolute value of the relative error from the three point extrapolation. As can be seen, the $h^2(1, 1/2, 1/4)$ values are extremely accurate. To achieve the same degree of accuracy without extrapolation, a much smaller mesh size would have to be used, which would have consumed much more computing time.

VI. CONTINUOUS VS DISCRETE SYSTEMS

From the structural mechanics point of view, all structural materials are considered to be continuous; although actual materials, from the microscopic point of view, consist of discrete particles, such as molecules or crystals. Based on the continuous model of material, differential or integral equations are derived. Since closed form exact solutions of these equations are usually difficult to obtain, especially when the structure is complicated, numerical methods must be used. In applying these numerical methods, the continuous structures are again decomposed into discrete systems. In the finite-element method, the structure is first divided into a finite number of discrete elements; equations governing these elements are then derived and solved. In the finite-difference method, the differential equations for the structures are first derived, the equations are next changed into algebraic finite-difference equations. A finite number of points is then chosen along the structures, and the finite-difference equations corresponding to these points are solved. In the method of characteristics, the partial differential equations are first transformed into ordinary differential equations in the characteristic coordinates. Next they are changed into finite-difference form, and a finite number of points are then chosen and the finite-difference equations solved.

It is interesting to note that the structural engineer may treat the structure as discrete or continuous system at his convenience. Care must be taken, however, to insure that the behavior of the corresponding system is compatible with the original system. The following example demonstrates a case where the continuous system and the corresponding discrete systems have drastically different behaviors.

The example to be considered is the longitudinal wave in a simple bar, as shown in Figure 6(a). The governing equation for the continuous bar is

$$u'' - \frac{1}{c^2} \ddot{u} = 0 \qquad (16)$$

where $c = \sqrt{E/\rho}$ is the bar velocity. This is a simple wave equation; all disturbances propagate at the velocity c with unchanged amplitude and shape. It is non-dispersive; if a sinusoidal solution

$$u = \sin \frac{2\pi}{\lambda} (x - c_p t) \qquad (17)$$

is substituted into (16), it can be shown readily that $c_p = c$, or the phase velocity, c_p, is independent of the wave length, λ.

Now, if we simulate the continuous bar by a mass-spring system, as shown in Figure 6(a), the governing equation of the system is

$$M \ddot{u}_p = k(u_{p+1} + u_{p-1} - 2 u_p) \qquad (18)$$

where u_p is the axial displacement of the pth mass particle, M is the mass of the particle, k is the spring constant. Substituting the sinusoidal solution

$$u_p = \sin \frac{2\pi}{\lambda} (x_p - c_p t) \qquad (19)$$

into (18), and keeping in mind that $x_p = pd$, we obtain

$$c_p{}^2 = \frac{d^2 k}{M} \frac{\sin^2(\pi ad)}{(\pi ad)^2} \qquad (20)$$

where $a = 1/\lambda$ is the wave number. From the relations $M = \rho Ad$, and $k = EA/d$, (20) becomes

$$c_p{}^2 = c^2 \frac{\sin^2(\pi ad)}{(\pi ad)^2} \qquad (21)$$

which indicates that the mass-spring system is dispersive; the phase velocity c_p is a function of wave length (or wave number). As can be seen from Figure 6, which is plotted from (21), when the wave length is very long as compared with d, the phase velocity c_p approaches c as a limit.

From this example we may conclude that in dividing a continuous system into a discrete system, we must make sure that the "representative length" of the discrete element be much smaller than the wave length to be encountered in the transient problem. (Another way of saying this is that the time required for disturbances to travel from one element to another must be much smaller than the time interval of interest in the transient problem). On the other hand, when smoothing out microscopic particles into a continuum, we must also make sure that the wave length is longer than the distances between particles. Indeed, the study of crystal structures by X-ray is based on the principle that the wavelength of X-rays is of the same order of magnitude as the interatomic spacing in a solid, (Ref. 17).

VII. COMPARISON WITH OTHER METHODS

The mode superposition method is suitable for the transient analysis of structures if the structure is not too complicated and if the loading is smooth, not involving step-inputs. An example of this method is a simply supported Timoshenko beam loaded at one end by a moment. Figure 7, which is reproduced from Figures 5 and 6 of Ref. 7, shows the time history of the moment at the center of the beam as calculated by two methods, an exact method (Laplace transform), and a modal method with static plus six modes. The results of the method of characteristics calculation also agree closely with the exact solution. Figure 7(a) shows the response due to a step moment input. As can be seen, the modal solution is not very accurate. Figure 7(b) shows the response for a ramp moment input where the modal solution is satisfactory.

The Laplace transform method is limited to very simple structures. Closed form solutions are seldom obtainable because of inversion difficulties. The Laplace transform method is very useful in producing exact solutions for certain problems which can be used as a standard in determining the accuracy of other approximate methods (Ref. 7).

The finite-difference method for one space variable problems is versatile; it can be applied to long time response problems where complicated discontinuities and wave reflections exist (Refs. 18, 19). In applying the finite-difference method, an artificial viscosity must be introduced to damp out spurious oscillations and to permit the numerical calculation of a discontinuous surface. As a result, sharp wave fronts cannot be maintained and the calculated stresses and velocities are not very accurate. Finite-difference methods for two space variable problems are currently being developed by many investigators (Ref. 20).

The finite-element method for static equilibrium problems is well developed. For transient problems, many new developments are also being made (Refs. 21, 22). The method is capable of handling more complicated structures; its stability and convergence behaviors are probably more difficult to establish.

The method of characteristics is suitable to treat almost all problems with one space variable. It is accurate, stable, and has good convergence. At the present, it cannot be used for two-dimensional transient structural problems, unless used in conjunction with some other methods.

VIII. CONCLUDING REMARKS

The discussion in this paper is limited to linear elastic structures. The method of characteristics is currently being extended in order to solve problems involving coupled second derivatives in the equations. Numerical schemes for problems with more than three equations are also being developed. Two space variable methods, as well as methods for plastic and viscoelastic materials will also be studied.

The method of characteristics is most suitable for comparatively simple structures. For very complicated structures, perhaps an approach combining the method of characteristics with the finite-element method may prove fruitful.

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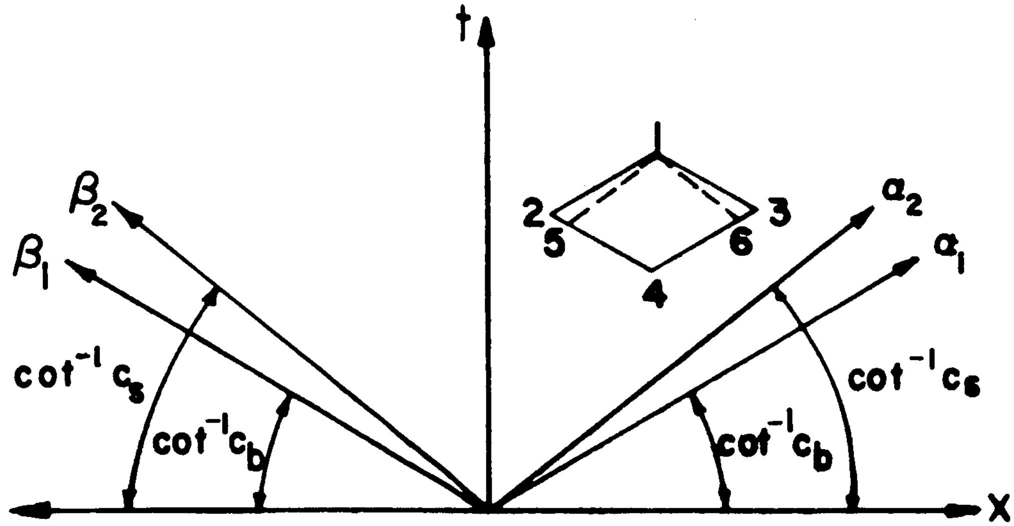


Figure 1
Characteristic Coordinates

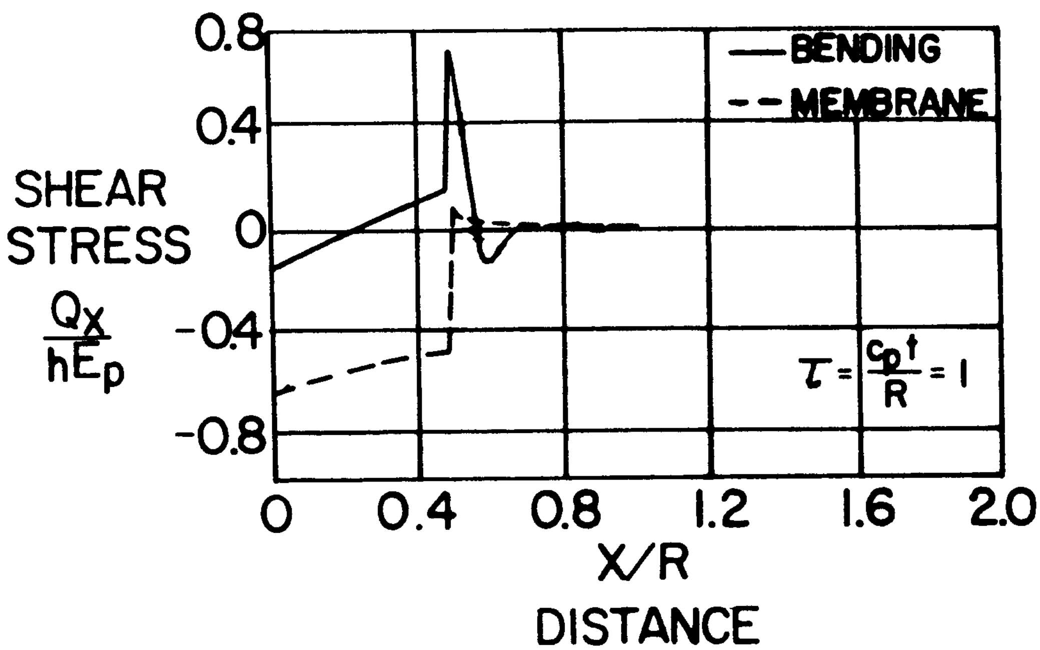


Figure 2

Shear Stress Distribution of a Cylindrical Shell Under a Load of $\dot{w}/c_p = 1$, $N_x = M_x = 0$ at $x = 0$; $\nu = 1/3$, $h/R = 0.1$, and $k^2 = 0.87$.

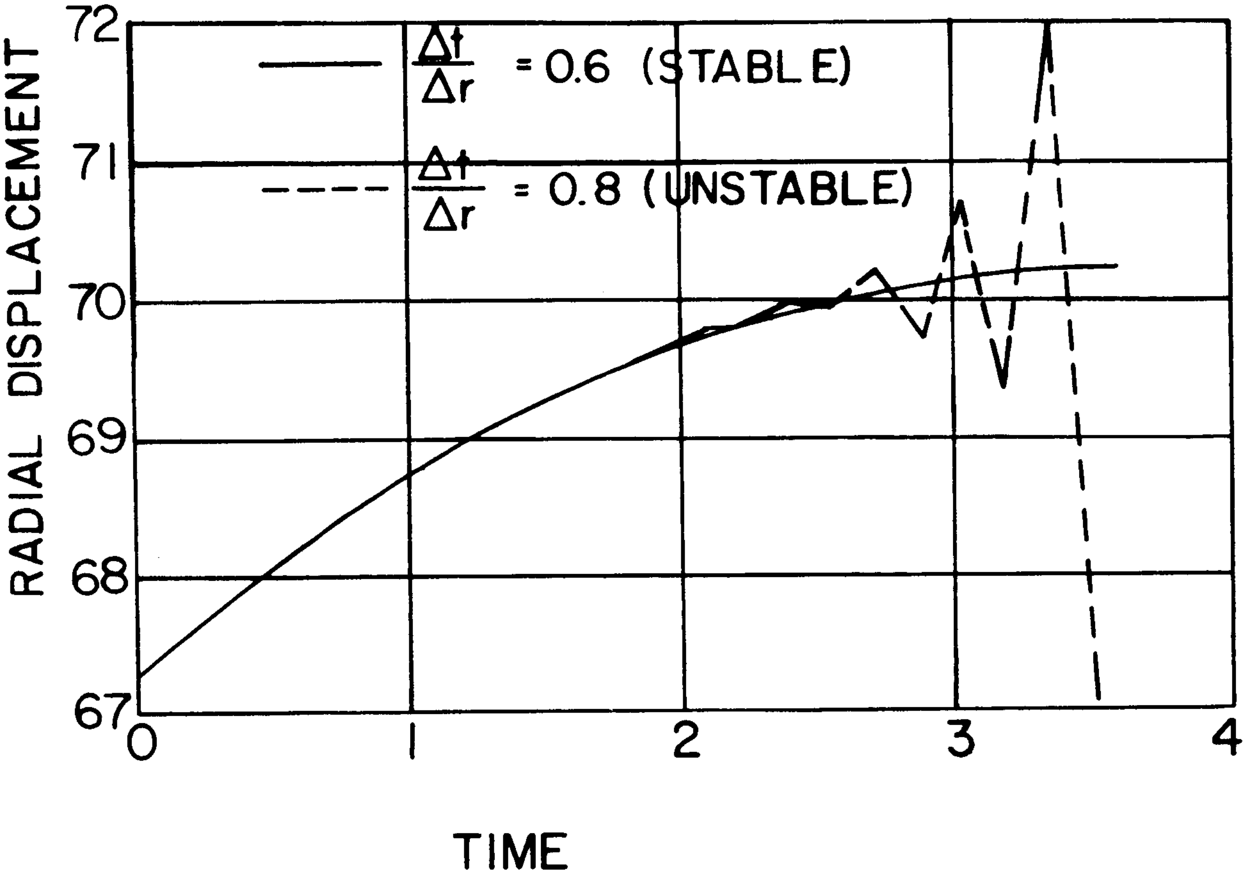


Figure 3

Stability of Numerical Methods - Calculated Time History of Displacement in Cylindrical Radial Direction of a Spherical Wave, by Two-Space Variable Method of Characteristics, at $r = z = 1.31$, $\Delta r = \Delta z = 0.02$.

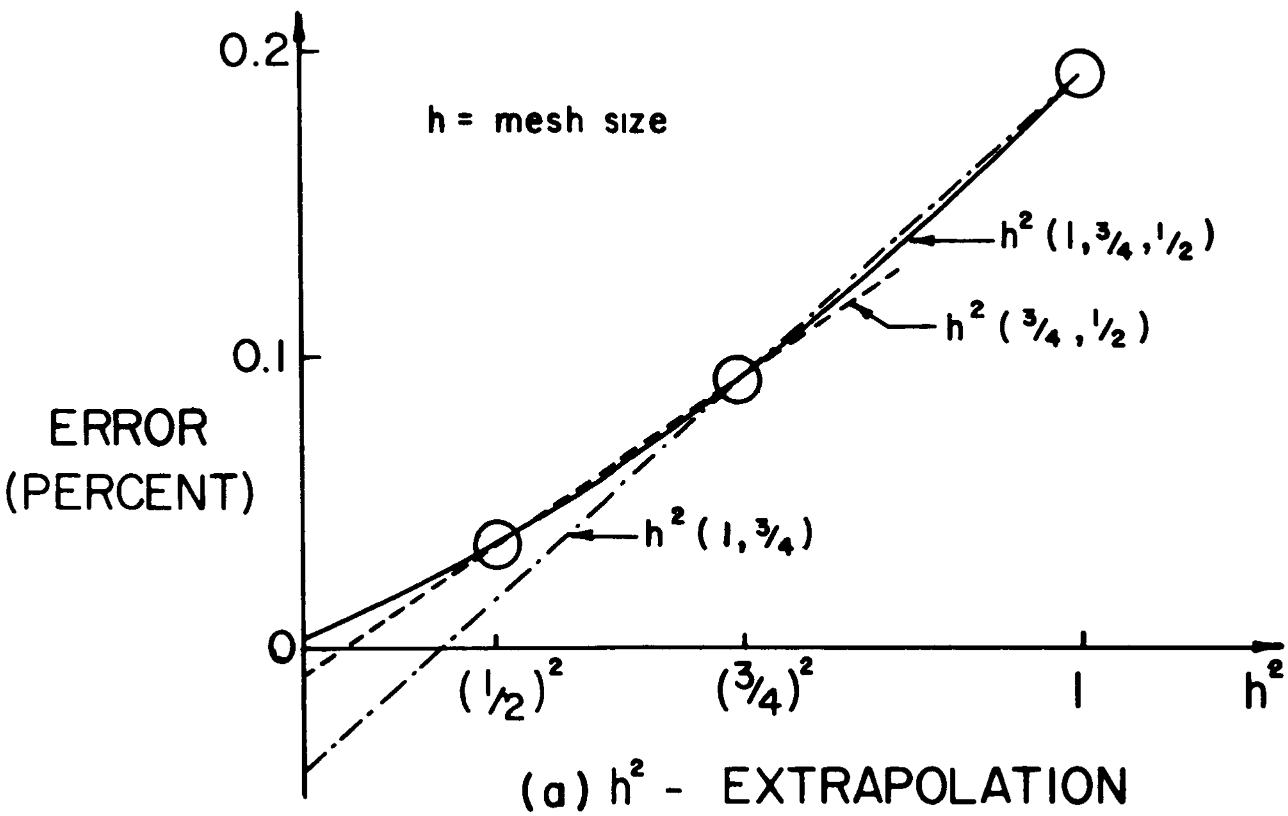


Figure 4(a)

Extrapolation of a Typical Set of h^2-type Error from Calculations with Mesh Sizes 1, 3/4, and 1/2.

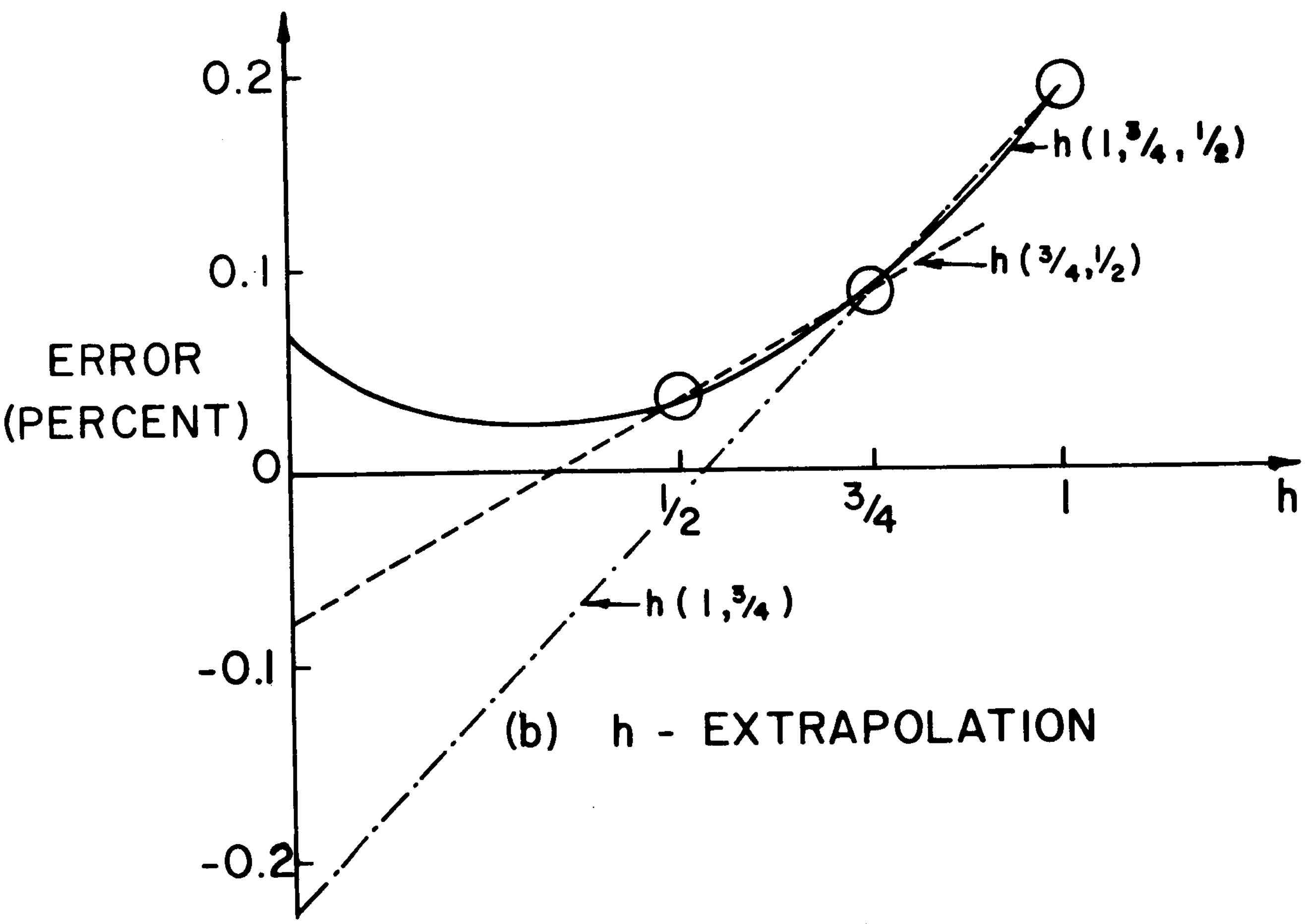


Figure 4(b)

Extrapolation of a Typical Set of h - type Error from Calculations with Mesh Sizes 1, 3/4, and 1/2.

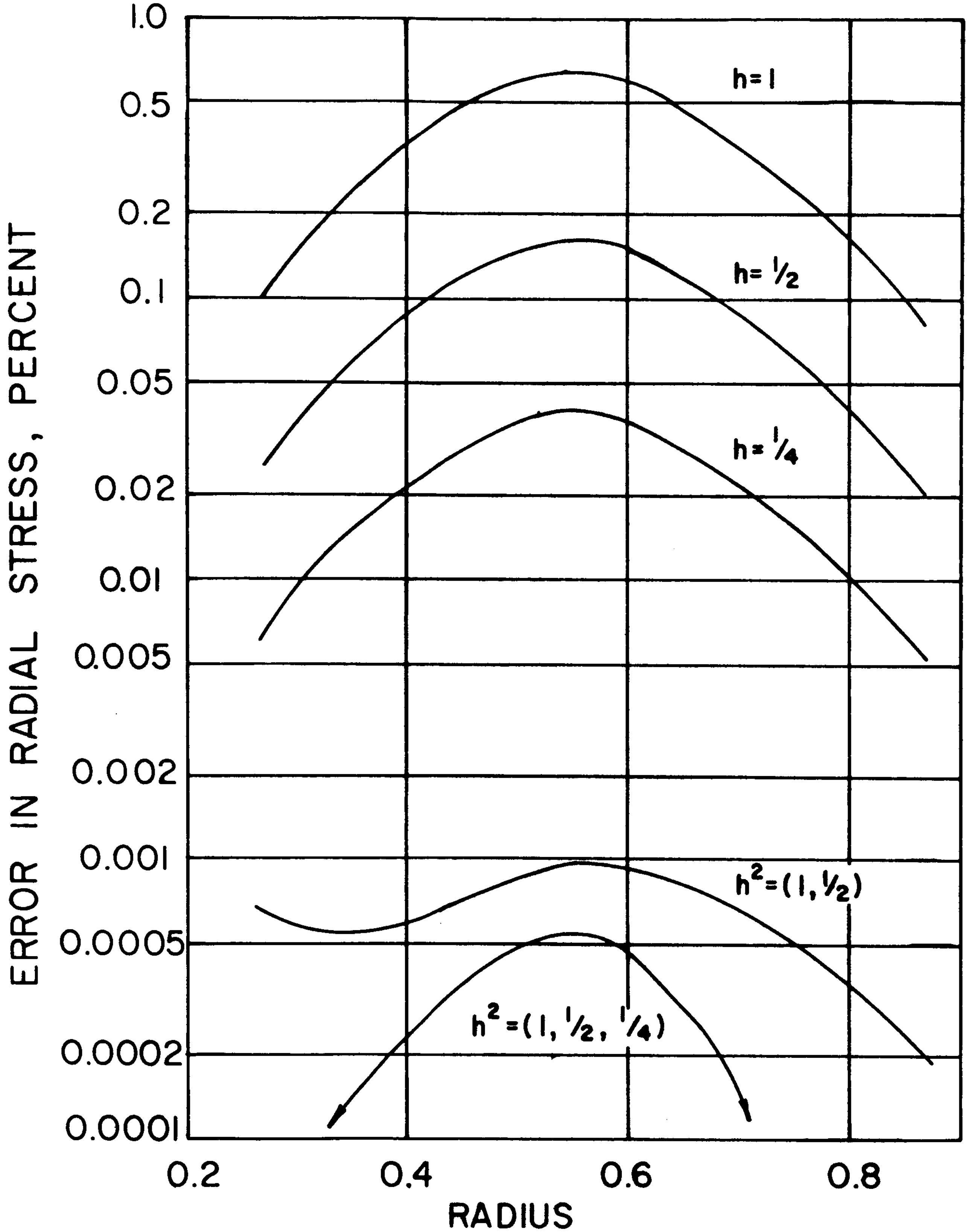
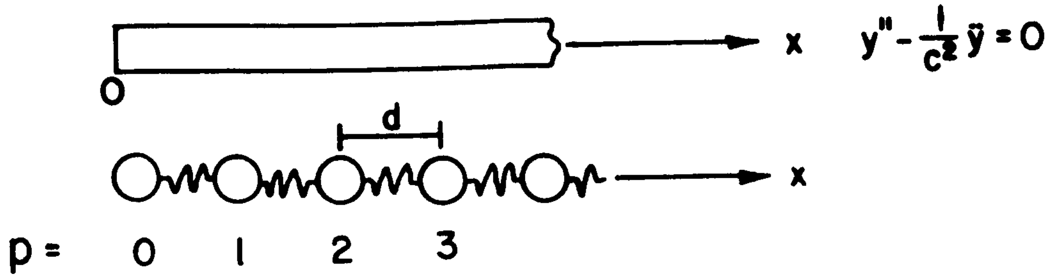


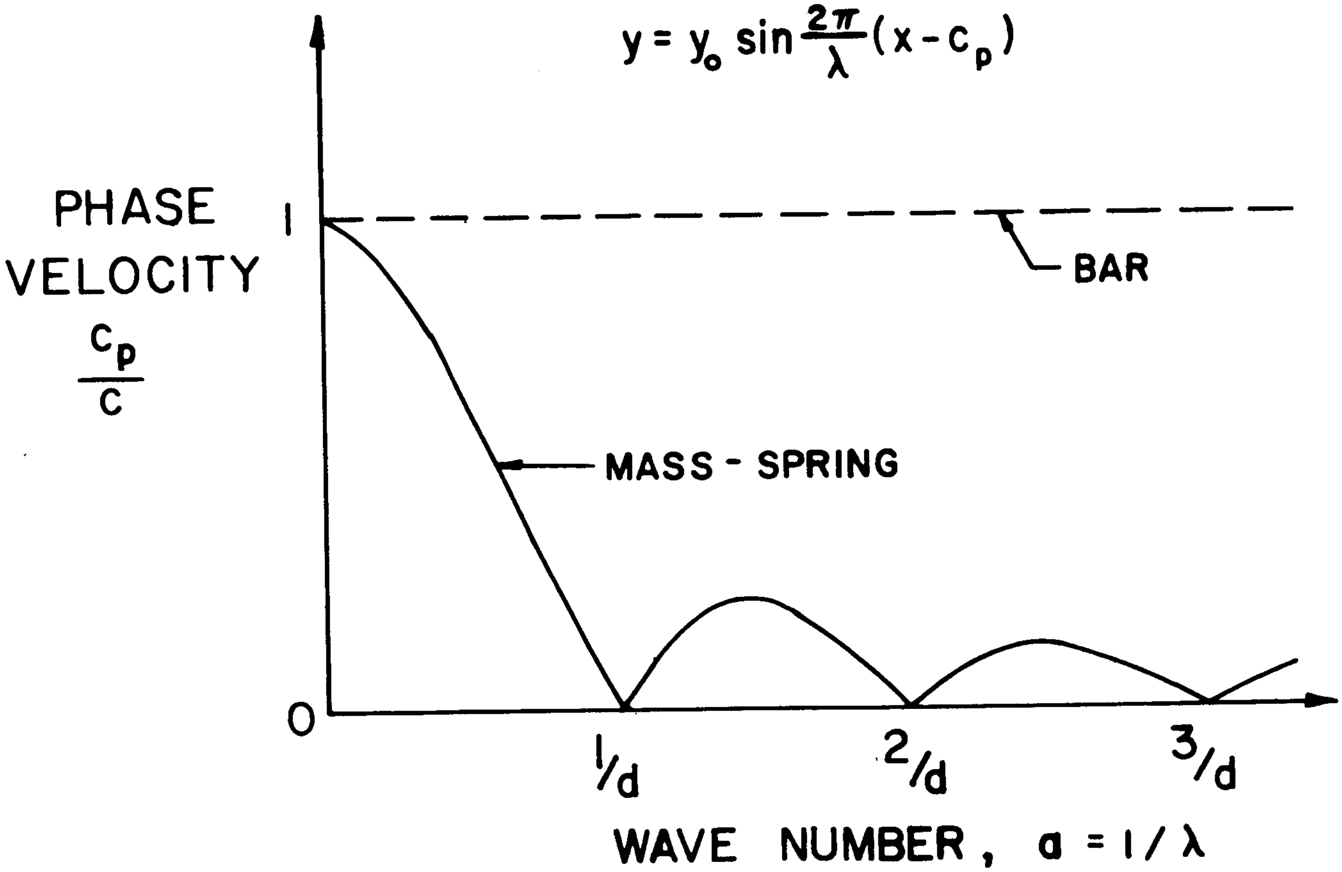
Figure 5

Percent Error in Calculated Radial Stress of a Spherical Dilatation Wave, with Mesh Sizes 1, 1/2, and 1/4, and the Absolute Value of the Extrapolated Errors.



(a)

A Continuous Bar and the Corresponding Mass-Spring System.



(b)

Phase Velocity versus Wave Number of the Bar and the Mass-Spring System

Figure 6

Dispersion of Longitudinal Waves in a Mass-Spring System, Simulating a Continuous Bar

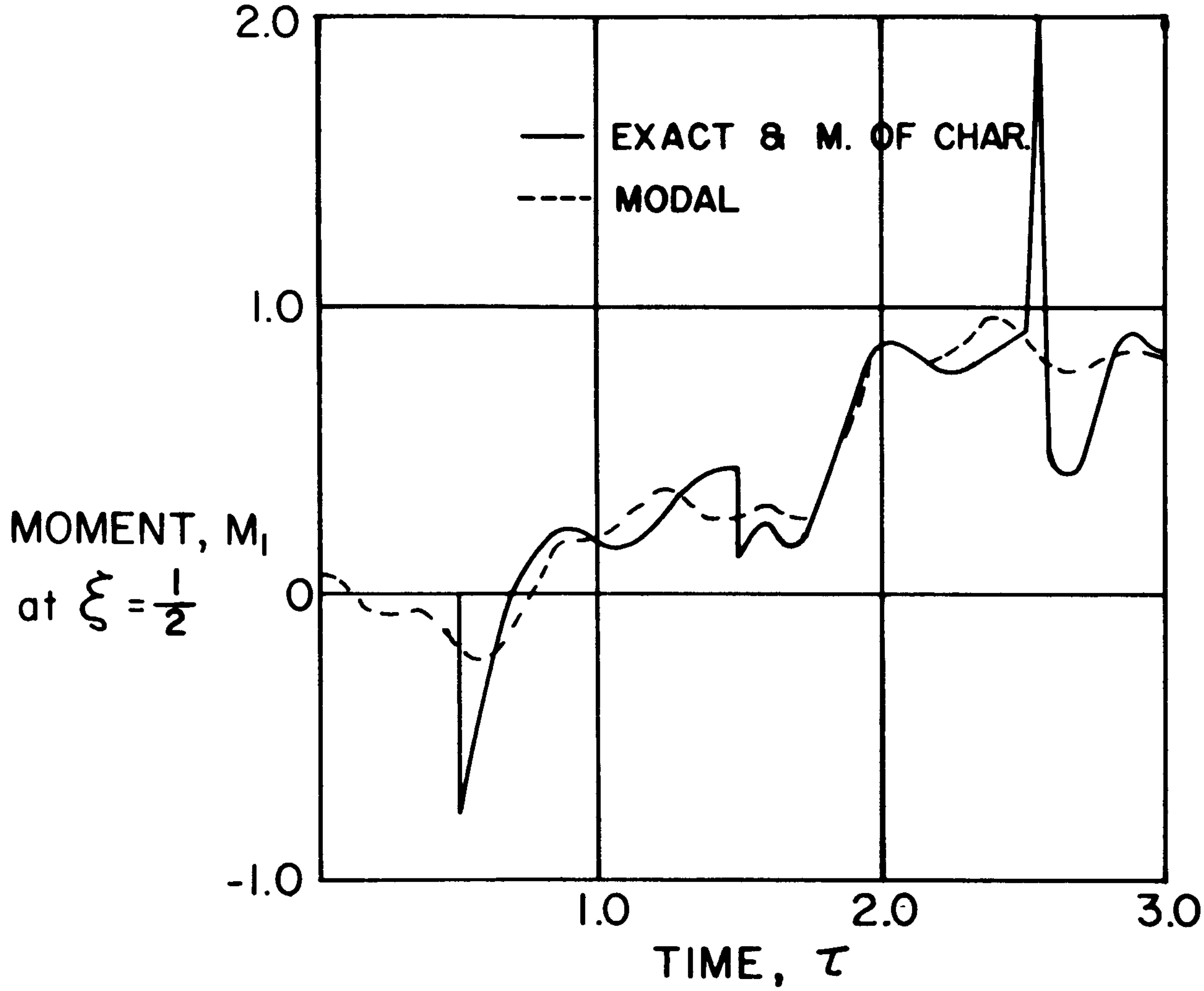


Figure 7(a)

Time History of Moment at the Center of a Uniform Simply Supported Beam.

(a) Beam Subjected to a Step Moment at $\xi = 0$

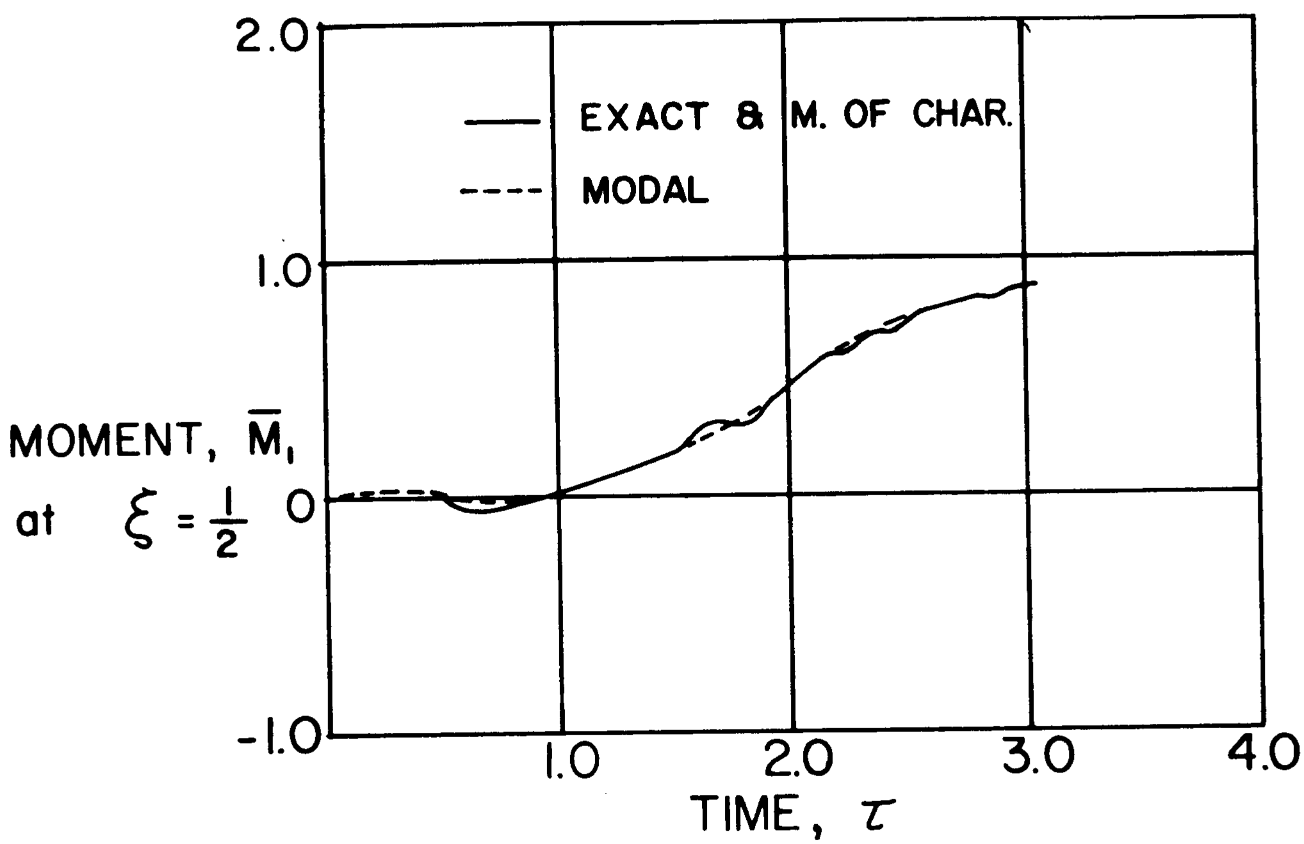


Figure 7(b)

Time History of Moment at the Center of a Uniform Simply Supported Beam.

(b) Beam Subjected to a Ramp-Platform Moment at $\xi = 0$.